

A. Standard exercises

12.1 Prove that if $\gamma : I \rightarrow S \subset \mathbb{R}^3$ is a C^2 biregular curve that is a geodesic of a co-oriented surface S , then the torsion of γ (as a curve in \mathbb{R}^3) coincides up to sign with the *geodesic torsion* of this curve. (This result explains the terminology “geodesic torsion”.) Is the converse statement true? (i.e. is a curve on a surface whose geodesic torsion equals its torsion always a geodesic?)

Hint. It is useful to compare the Darboux equations with the Serret–Frenet equations for the curve γ .

12.2 Consider the circular right cylinder

$$C \subset \mathbb{R}^3, \quad x^2 + y^2 = a^2.$$

Compute its Gauss and mean curvature (say with respect to the co-orientation corresponding to the inwards pointing normal). Can you predict these values without performing calculations?

12.3 Consider the helicoid surface, which is parametrized by

$$\psi(u, v) = (v \cos u, v \sin u, u).$$

Compute the metric tensor, the second fundamental form, and the Weingarten map. What are the mean curvature and Gauss curvature of this surface?

12.4 Let S be the graph (in \mathbb{R}^3) of the C^2 function

$$\varphi : \Omega \rightarrow \mathbb{R}, \quad \Omega \subset \mathbb{R}^2.$$

Show that the matrix of the second fundamental form is

$$H = \frac{1}{\sqrt{1 + \varphi_x^2 + \varphi_y^2}} \begin{pmatrix} \varphi_{xx} & \varphi_{xy} \\ \varphi_{xy} & \varphi_{yy} \end{pmatrix}.$$

12.4 Let $S \subset \mathbb{R}^3$ be a C^2 surface and $p \in S$. Let $L_p : T_p S \rightarrow T_p S$ be the shape operator at p . As we have seen in class, L_p is self-adjoint (recall the definition of this notion). Is the matrix of L_p (associated to a local parametrization $\circ\psi : \Omega \rightarrow S$ of S around p) always symmetric?

12.6 In this exercise, we are going to adopt the notations of Ex. 9.5 regarding ruled surfaces: Let $S \subset \mathbb{R}^3$ be a C^2 ruled surface parametrized by $\psi : I \times (-\epsilon, \epsilon) \rightarrow S$,

$$\psi(u, v) = \gamma(u) + b(u)v,$$

where $\gamma : I \rightarrow \mathbb{R}^3$ is a C^2 regular curve and $b : I \rightarrow \mathbb{R}^3$ is a non-zero C^2 vector field along γ .

- (a) Show that the Gauss curvature of S satisfies $K \leq 0$.
- (b) Show that $K = 0$ when $\dot{\gamma}, b, \dot{b}$ are linearly dependent. Verify that this is indeed the case when S is a conical surface (see Ex. 11.2).

Remark. Note that the surfaces in Ex. 12.2 and 12.3 are ruled surfaces and, hence, the above result applies to them.

B. Bonus exercise (on geodesics of surfaces of revolution)

12.7 The goal of this exercise is to determine all geodesics of surfaces of revolution. Consider the surface of revolution $\psi : \Omega = [0, 2\pi] \times I \rightarrow S \subset \mathbb{R}^3$ given by

$$\psi(\theta, s) = (r(s) \cos \theta, r(s) \sin \theta, z(s)).$$

- (a) Let $\gamma(t)$ ($t \in J$) be a C^2 curve traced on the surface S . Show that for all $t \in J$,

$$\langle \ddot{\gamma}(t), \frac{\partial \psi}{\partial \theta} \rangle = \frac{d}{dt} (r^2(t) \dot{\theta}(t)),$$

where a dot denotes differentiation with respect to the parameter t .

- (b) Show that if $\gamma(t)$ is a geodesic, then the quantities

$$r^2(t) \dot{\theta}(t) \quad \text{and} \quad (r(t) \dot{\theta}(t))^2 + \dot{r}(t)^2 + \dot{z}(t)^2$$

are constant (independent of t).

Remark. The first of the above “conservation laws” is related to the fact that translations in θ (i.e. rotations of the surface) are isometric motions, i.e. preserve the geometry of the surface. This can be seen as a special case of the more general *Noether’s theorem* for equations arising from a Lagrangian, applied to the case of the equations satisfied by a geodesic. The second conservation law is simply the fact that $\|\dot{\gamma}\|$ is constant for a geodesic.

- (c) Show that the function $t \mapsto 1/r(t)$ is bounded for any geodesic that is not a meridian of the surface of revolution.